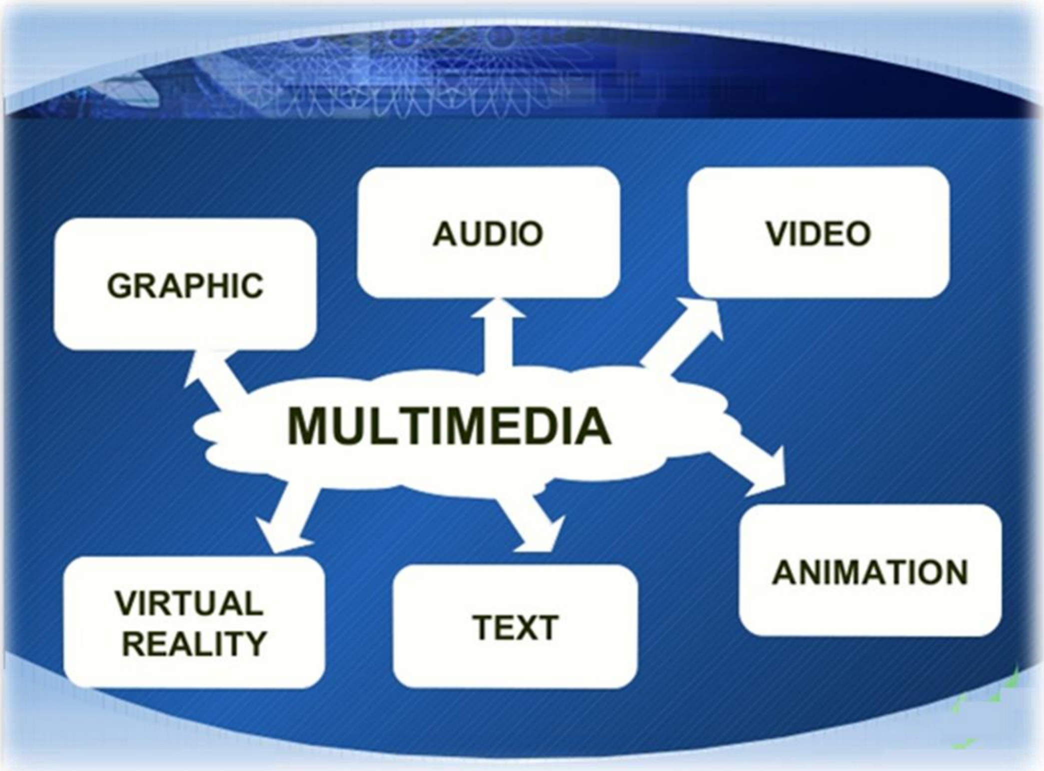

Tikrit University

Computer Science Dept.

Master Degree

Lecture 9

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Discrete Foureir Transform

Any periodic function can be expressed in a Fourier series representation. The discrete-time Fourier transform (DTFT) $X(\omega)$ of a discrete-time sequence $x(n)$ is a periodic continuous function of ω with a period of 2π. So it cannot be processed by a digital system. For processing by a digital system it should be converted into discrete form. The DFT converts the continuous function of ω to a discrete function of ω. Thus, DFT allows us to perform frequency analysis on a digital computer.

The DFT of a discrete-time signal $x(n)$ is a finite duration discrete frequency sequence. The DFT sequence is denoted by $X(k)$. The DFT is obtained by sampling one period of the Fourier transform $X(\omega)$ of the signal $x(n)$ at a finite number of frequency points. This sampling is conventionally performed at N equally spaced points in the period $0 \le \omega \le 2\pi$ or at $\omega_k = 2\pi k/N$; $0 \le k \le N - 1$. We can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite length into discrete frequency sequence $X(k)$ of finite length.

The DFT is important for two reasons. First it allows us to determine the frequency content of a signal, that is to perform spectral analysis. The second application of the DFT is to perform filtering operation in the frequency domain.

Let $x(n)$ be a discrete-time sequence with Fourier transform $X(\omega)$, then the DFT of $x(n)$ denoted by $X(k)$ is defined as:

$$X(k) = X(\omega)\big|_{\omega=(2\pi k/N)}; \quad \text{for } k = 0, 1, 2, \dots, N-1$$

The DFT of $x(n)$ is a sequence consisting of N samples of $X(k)$. The DFT sequence starts at $k = 0$, corresponding to $\omega = 0$, but does not include $k = N$ corresponding to $\omega = 2\pi$ (since the sample at $\omega = 0$ is same as the sample at $\omega = 2\pi$).

To calculate the DFT of a sequence, it is not necessary to compute its Fourier transform, since the DFT can be directly computed.

DFT The N-point DFT of a finite duration sequence $x(n)$ of length L, where $N \ge L$ is defined as:

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)W_N^{nk}; \quad \text{for } k = 0, 1, 2, \dots, N-1$$

IDFT The Inverse Discrete Fourier transform (IDFT) of the sequence $X(k)$ of length N is defined as:

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}; \quad \text{for } n = 0, 1, 2, \dots, N-1$$

where $W_N = e^{-j(2\pi/N)}$ is called the twiddle factor.

The N-point DFT pair $x(n)$ and $X(k)$ is denoted as:

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

Example 1 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.
(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(k)$ to be found is N = 4-point and length of the sequence $L = 4$. So no padding of zeros is required.
We know that the DFT $\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{3} x(n)\, e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = \sum_{n=0}^{3} x(n)\, e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1)\, e^{-(j\pi/2)} + x(2)\, e^{-j\pi} + x(3)\, e^{-j(3\pi/2)}$$

$$= 1 + (-1)(0 - j) + 2(-1 - j0) - 2(0 + j)$$

$$= -1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 - 1(-1 - j0) + 2(1 - j0) - 2(-1 - j0) = 6$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j(3\pi/2)n} = x(0) + x(1) e^{-j(3\pi/2)} + x(2) e^{-j3\pi} + x(3) e^{-j(9\pi/2)}$$

$$= 1 - 1(0 + j) + 2(-1 - j0) - 2(0 - j) = -1 + j$$

$$\therefore \quad X(k) = \{0, -1 - j, 6, -1 + j\}$$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk}$$

i.e. $$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(\pi/2)nk}$$

$$\therefore \quad x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{0} = \frac{1}{4} [X(0) + X(1) + X(2) + X(3)]$$

$$= \frac{1}{4} [4 + 2 + 0 + 4] = 2.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(\pi/2)k} = \frac{1}{4} \left[X(0) + X(1) e^{j(\pi/2)} + X(2) e^{j\pi} + X(3) e^{j(3\pi/2)} \right]$$

$$= \frac{1}{4}[4 + 2(0 + j) + 0 + 4(0 - j)] = 1 - j0.5$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[4 + 2(-1 + j0) + 0 + 4(-1 + j0)] = -0.5$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[4 + 2(0 - j) + 0 + 4(0 + j)] = 1 + j0.5$$

$$x_3(n) = \{2.5,\ 1 - j0.5,\ -0.5,\ 1 + j0.5\}$$

Example 2 (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.
Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{3} x(n)e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)n} = x(0) + x(1)e^{-j(\pi/2)} + x(2)e^{-j\pi} + x(3)e^{-j(3\pi/2)}$$

$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n} = x(0) + x(1)e^{-j(3\pi/2)} + x(2)e^{-j3\pi} + x(3)e^{-j(9\pi/2)}$$

$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{0} = \frac{1}{4}[X(0) + X(1) + X(2) + X(3)] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(\pi/2)} + X(2)e^{j\pi} + X(3)e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[1+0+e^{j\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1+0+e^{j2\pi}+0] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[1+0+e^{j3\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$\therefore$ The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

Example 3 Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT}\, x(n) = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{2} x(n)e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1)e^{-j(2\pi/3)k} + x(2)\,e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k\right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k\right)$$

When $k = 0$, $\quad X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $\quad X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$
$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

To compute the 6-point DFT, convert the 3-point sequence $x(n)$ into 6-point sequence by padding with zeros.

$$x(n) = \{2, 1, 2, 0, 0, 0\}, \quad N = 6$$

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{5} x(n)e^{-j(2\pi/N)nk}, \quad k = 0, 1, 2, 3, 4, 5$$
$$= x(0) + x(1)e^{-j(2\pi/6)k} + x(2)e^{-j(4\pi/6)k} + x(3)e^{-j(6\pi/6)k} + x(4)e^{-j(8\pi/6)k} + x(5)e^{-j(10\pi/6)k}$$
$$= 2 + e^{-j(\pi/3)k} + 2e^{-j(2\pi/3)k}$$

When $k = 0$, $X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(1) = 2 + e^{-j(\pi/3)} + 2e^{-j(2\pi/3)}$

$$= 2 + (0.5 - j0.866) + 2(-0.5 - j0.866) = 1.5 - j2.598$$

When $k = 2$, $X(2) = 2 + e^{-j(2\pi/3)} + 2e^{-j(4\pi/3)}$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866) = 0.5 + j0.866$$

When $k = 3$, $X(3) = x(0) + x(1)e^{-j(3\pi/3)} + x(2)e^{-j(6\pi/3)}$

$$= 2 + (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi)$$
$$= 2 - 1 + 2 = 3$$

When $k = 4$, $X(4) = x(0) + x(1)e^{-j(4\pi/3)} + x(2)e^{-j(8\pi/3)}$

$$= 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$$
$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$
$$= 0.5 - j0.866$$

When $k = 5$, $X(5) = x(0) + x(1)e^{-j(5\pi/3)} + x(2)e^{-j(10\pi/3)}$

$$= 2 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$
$$= 2 + (0.5 - j0.866) + 2(-0.5 + j0.866) = 1.5 + j0.866$$

Tabulating the above 3-point and 6-point DFTs, we have

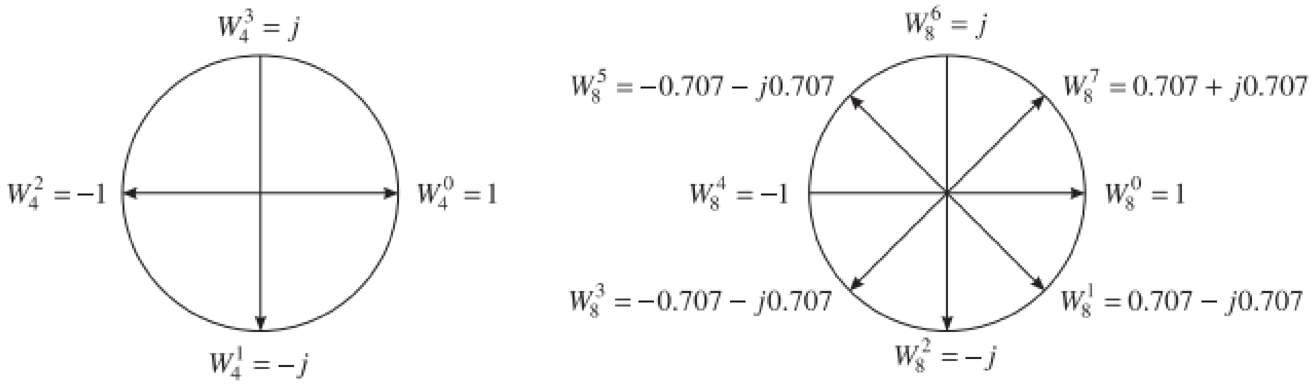
DFT	$X(0)$	$X(1)$	$X(2)$	$X(3)$	$X(4)$	$X(5)$
3-point	5	$0.5 + j0.866$	$0.5 - j0.866$	–	–	–
6-point	5	$1.5 - j2.598$	$0.5 + j0.866$	3	$0.5 - j0.866$	$1.5 + j0.866$

Since 6-point = 3 × 2-point
$X(k)$ of 3-point sequence = $X(2k)$ of 6-point sequence

A SLIGHTLY FASTER METHOD FOR COMPUTING DFT VALUES

The computation procedure given above is very lengthy and cumbersome. We can improve the situation somewhat by using the powers of twiddle factor (W_N) instead of the factors of $e^{-j(2\pi/N)}$. This procedure may be thought of as the forerunner of the method, using the Fast Fourier Transformation (FFT) algorithm.

As stated above, we use the powers of W_N in the multiplications instead of $e^{-j(2\pi/N)}$ and its factors. Figure 1 and Table 1 show the powers of W for various DFTs.



(a) 4-point DFT (b) 8-point DFT

Figure 1: Values of W_N^n for 4-point and 8-point DFTs

Table 1 showing powers of the factor W_N for 4-point and 8-point DFTs

Twiddle factor	*4-point DFT*	*Twiddle factor*	*8-point DFT*
W_4^0	1	W_8^0	1
W_4^1	$-j$	W_8^1	$0.707 - j0.707$
W_4^2	-1	W_8^2	$-j$
W_4^3	j	W_8^3	$-0.707 - j0.707$
$W_4^4 = W_4^0$	1	W_8^4	-1
$W_4^5 = W_4^1$	$-j$	W_8^5	$-0.707 - j0.707$
$W_4^6 = W_4^2$	-1	W_8^6	j
$W_4^7 = W_4^3$	j	W_8^7	$0.707 + j0.707$

Example 4 Obtain DFT of the sequence $x(n) = \{x(0), x(1), x(2), x(3)\} = \{1, 0, -1, 2\}$.

Solution: Given $x(n) = \{x(0), x(1), x(2), x(3)\} = \{1, 0, -1, 2\}$

Here $N = 4$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{3} x(n)W_4^{nk}$$

$$= x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}, \qquad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0 = 1 + 0 - 1 + 2 = 2$$

$$X(1) = x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3 = 1 + 0 + (-1)(-1) + 2j = 2 + j2$$

$$X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6 = 1 + 0 + (-1)(1) + 2(-1) = -2$$

$$X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9 = 1 + 0 + (-1)(-1) + 2(-j) = 2 - j2$$

We get the DFT sequence as $X(k) = \{2, 2 + j2, -2, 2 - j2\}$.

Thus, we find that our new method is faster than the direct method of evaluating the DFT.

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

THE IDFT FROM THE MATRIX FORM

The matrix **x** may be expressed in terms of the inverse of $\mathbf{W_N}$ as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\ \mathbf{x}$$

The matrix $\mathbf{W_N^{-1}}$ is called the IDFT matrix. We may also obtain **x** directly from the IDFT relation in matrix form, where the change of index from n to k and the change in the sign of the exponent in $e^{j(2\pi/N)nk}$ lead to the conjugate transpose of $\mathbf{W_N}$. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

Comparison of the two forms suggests that $\mathbf{W_N^{-1}} = \frac{1}{N}\left[\mathbf{W_N^*}\right]^T$.

This very important result shows that $\mathbf{W_N^{-1}}$ requires only conjugation and transposition of $\mathbf{W_N}$, an obvious computational advantage.

USING THE DFT TO FIND THE IDFT

Both the DFT and IDFT are matrix operations and there is an inherent symmetry in the DFT and IDFT relations. In fact, we can obtain the IDFT by finding the DFT of the conjugate sequence and then conjugating the results and dividing by N, Mathematically,

$$x(n) = \text{IDFT}[X(k)] = \frac{1}{N}[\text{DFT}\{X^*(k)\}]^*$$

This result involves the conjugate symmetry and duality of the DFT and IDFT, and suggests that the DFT algorithm itself can also be used to find the IDFT. In practice, this is indeed what is done.

Example 5 Find the DFT of the sequence $x(n) = \{1, 2, 1, 0\}$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

Example 6 Find the DFT of $x(n) = \{1, -1, 2, -2\}$.

Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k)=\begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$\therefore$ $\text{DFT}\{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$

Example 7 Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

Solution: Given $x(n) = \{1, -2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix as shown below.

$$\text{DFT}\{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$\therefore$ $\text{DFT}\{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$

Example 8 Find the 8-point DFT of $x(n) = \{1, 1, 0, 0, 0, 0, 0, 0\}$. Use the property of conjugate symmetry.

Solution: For given $x(n) = \{1, 1, 0, 0, 0, 0, 0, 0\}$,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{7} x(n)\, e^{-j(2\pi/8)nk}$$

$$= 1 + e^{-j(\pi/4)k}, \qquad k = 0, 1, 2, \ldots, 7$$

Since $N = 8$, we need to compute $X(k)$ only for $k \le (8/2) = 4$.

$$X(0) = 1 + 1 = 2$$

$$X(1) = 1 + e^{-j(\pi/4)} = 1 + 0.707 - j0.707 = 1.707 - j0.707$$

$$X(2) = 1 + e^{-j(\pi/2)} = 1 + 0 - j = 1 - j$$

$$X(3) = 1 + e^{-j(3\pi/4)} = 1 - 0.707 - j0.707 = 0.293 - j0.707$$

$$X(4) = 1 + e^{-j\pi} = 1 - 1 = 0$$

With N = 8, conjugate symmetry says $X(k) = X^*(N - k) = X^*(8 - k)$ and we find

$$X(5) = X^*(8-5) = X^*(3) = 0.293 + j0.707$$

$$X(6) = X^*(8-6) = X^*(2) = 1 + j$$

$$X(7) = X^*(8-7) = X^*(1) = 1.707 + j0.707$$

Thus, $X(k) = \{2, 1.707 - j0.707, 1 - j, 0.293 - j0.707, 0, 0.293 + j0.707, 1 + j, 1.707 + j0.707\}$

Example 9 Find the IDFT of $X(k)=\{4, -j2, 0, j2\}$ using DFT.

Solution: Given $X(k) = \{4, -j2, 0, j2\}$ $\therefore$ $X^*(k) = \{4, j2, 0, -j2\}$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\,\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

Example 10 Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ using DFT.

Solution: Given $X(k) = \{4, 2, 0, 4\}$

$\therefore$ $X^*(k) = \{4, 2, 0, 4\}$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\,[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

Example 11 Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution: Given $X(k) = \{1, 0, 1, 0\}$, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrix as shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\,\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\,\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$