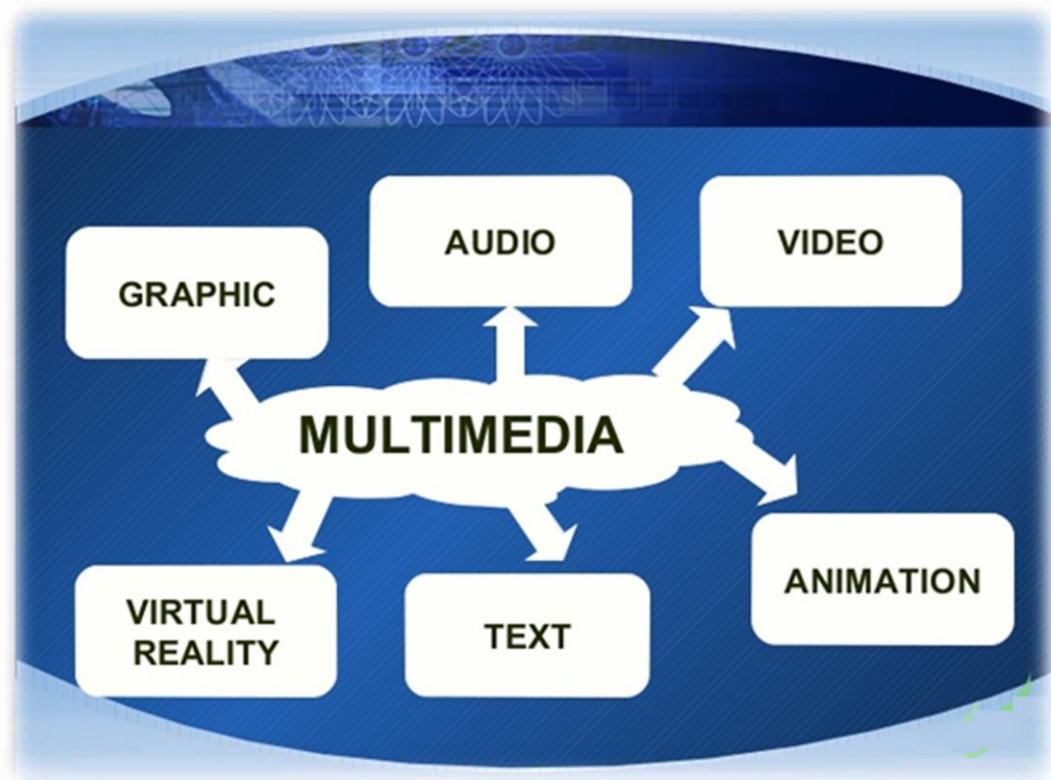




Tikrit University
Computer Science Dept.
Master Degree
Lecture 8

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Convolution

Convolution is a mathematical way of combining two signals to form a third signal. It is the single most important technique in Digital Signal Processing. Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

A discrete-time system performs an operation on an input signal based on a predefined criterion to produce a modified output signal. The input signal $x(n)$ is the system excitation and the output signal $y(n)$ is the system response. This transform operation is shown in Figure 1.



Figure 1 A discrete-time system.

If the input to the system is a unit impulse, i.e. $x(n) = \delta(n)$, then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)]$$

The system is assumed to be initially relaxed, i.e. the system has zero initial conditions.

We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Example 1

Consider the special case of a finite duration sequence given as

$x(n) = \left\{ -2, \underset{\uparrow}{4}, 0, 3 \right\}$ Resolve the sequence $x(n)$ into a sum of weighted impulse response

Solution

Since the sequence $x(n)$ is non-zero for the time instants $n = -1, 0, 1, 2$, we need impulses at delay $k = -1, 0, 1, 2$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$

Using the above equation, we get

$$x(n) = 2\delta(n+1) + 4\delta(n) + 0\delta(n-1) + 3\delta(n-2).$$

So the system response $y(n)$ is given by

$$y(n) = T[x(n)] = T \left[\sum_{k=-\infty}^{\infty} x(k) \delta(n-k) \right]$$

For a linear system, the above equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) T[\delta(n-k)]$$

The response due to shifted impulse sequence $\delta(n-k)$ can be denoted by $h(n, k)$, i.e.

$$h(n, k) = T[\delta(n-k)]$$

For a shift-invariant system,

Delayed output = Output due to delayed input

i.e.
$$h(n-k) = h(n, k)$$

Therefore,
$$T[\delta(n-k)] = h(n-k)$$

Therefore, the equation for $y(n)$ reduces to

Therefore, the equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

So we can conclude that:

For a linear time invariant system, if the input sequence $x(n)$ and the impulse response $h(n)$ are given, the output sequence $y(n)$ can be found using the equation:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

This is known as convolution sum and is represented as

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

where $*$ denotes the convolution operation.

This is an extremely powerful and useful result that allows us to compute the system output for any input signal excitation.

Properties of convolution

1. Commutative property: $x(n) * h(n) = h(n) * x(n)$
2. Associative property: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3. Distributive property: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
4. Shifting property:
If $x(n) * h(n) = y(n)$, then $x(n-k) * h(n-m) = y(n-k-m)$
5. Convolution with an impulse: $x(n) * \delta(n) = x(n)$

CONVOLUTION OF FINITE SEQUENCES

In practice, we often deal with sequences of finite length, and their convolution may be found by several methods. The convolution $y(n)$ of two finite-length sequences $x(n)$ and $h(n)$ is also of finite length and is subject to the following rules, which serve as useful consistency checks:

1. The starting index of $y(n)$ equals the sum of the starting indices of $x(n)$ and $h(n)$.
2. The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ and $h(n)$.
3. The length L_y of $y(n)$ is related to the lengths L_x and L_h of $x(n)$ and $h(n)$ by $L_y = L_x + L_h - 1$.

Method 1 Linear Convolution Using Graphical Method

- Step 1:* Choose the starting time n for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$ is a good choice.
- Step 2:* Express both the sequences $x(n)$ and $h(n)$ in terms of the index k .
- Step 3:* Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by n to the right if n is positive and to the left if n is negative to obtain $h(n - k)$.
- Step 4:* Multiply the two sequences $x(k)$ and $h(n - k)$ element by element and sum the products to get $y(n)$.
- Step 5:* Increment the index n , shift the sequence $h(n - k)$ to the right by one sample and perform Step 4.
- Step 6:* Repeat Step 5 until the sum of products is zero for all remaining values of n .

Method 2 Linear Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given N sample sequences. Let $x_3(n)$ be the N sample sequence obtained by linear convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

- Step 1:* Change the index from n to k , and write $x_1(k)$ and $x_2(k)$.
- Step 2:* Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.
- Step 3:* Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.
- Step 4:* Shift the sequence $x_2(-k)$, q times to get the sequence $x_2(q - k)$. If q is positive, then shift the sequence to the right and if q is negative, then shift the sequence to the left.
- Step 5:* The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k) x_2(q - k)$$

Determine the product sequence $x_1(k)x_2(q - k)$ for one period.

- Step 6:* The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 3 Linear Convolution Using Tabular Method

Given $x(n) = \{x_1, x_2, x_3, x_4\}$, $h(n) = \{h_1, h_2, h_3, h_4\}$

The convolution of $x(n)$ and $h(n)$ can be computed as per the following procedure.

Step 1: Write down the sequences $x(n)$ and $h(n)$ as shown in Table 2.1.

Step 2: Multiply each and every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.

Step 3: Group the elements in the table by drawing diagonal lines as shown in table.

Step 4: Starting from the left sum all the elements in each strip and write down in the same order.

$$y(n) = x_1h_1, x_1h_2 + x_2h_1, x_1h_3 + x_2h_2 + x_3h_1, x_1h_4 + x_2h_3 + x_3h_2 \\ + x_4h_1, x_2h_4 + x_3h_3 + x_4h_2, x_3h_4 + x_4h_3, x_4h_4$$

Step 5: Mark the symbol $\uparrow$ at time origin ($n = 0$).

TABLE 1 Table for Computing $y(n)$

		$x(n)$			
		x_1	x_2	x_3	x_4
$h(n)$	h_1	x_1h_1	x_2h_1	x_3h_1	x_4h_1
	h_2	x_1h_2	x_2h_2	x_3h_2	x_4h_2
	h_3	x_1h_3	x_2h_3	x_3h_3	x_4h_3
	h_4	x_1h_4	x_2h_4	x_3h_4	x_4h_4

Method 4 Linear Convolution Using Matrices

If the number of elements in $x(n)$ are N_1 and in $h(n)$ are N_2 , then to find the convolution of $x(n)$ and $h(n)$ form the following matrices:

1. Matrix H of order $(N_1 + N_2 - 1) \times N_1$ with the elements of $h(n)$
2. A column matrix X of order $(N_1 \times 1)$ with the elements of $x(n)$
3. Multiply the matrices H and X to get a column matrix Y of order $(N_1 + N_2 - 1)$ that has the elements of $y(n)$, the convolution of $x(n)$ and $h(n)$.

$$\begin{bmatrix}
 h(0) & 0 & \cdots & 0 \\
 h(1) & h(0) & \cdots & 0 \\
 \vdots & \vdots & \cdots & 0 \\
 h(N_2 - 1) & h(N_2 - 2) & \cdots & h(0) \\
 0 & h(N_2 - 1) & \cdots & h(1) \\
 \vdots & \vdots & \cdots & \vdots \\
 0 & 0 & \cdots & h(N_2 - 1)
 \end{bmatrix}
 \begin{bmatrix}
 x(0) \\
 x(1) \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 x(N_1 - 1)
 \end{bmatrix}
 =
 \begin{bmatrix}
 y(0) \\
 y(1) \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 y(N_1 + N_2 - 1)
 \end{bmatrix}$$

$H \qquad \qquad \qquad X \qquad = \qquad Y$

EXAMPLE 2 Determine the convolution sum of two sequences:

$$x(n) = \{4, 2, 1, 3\}, \quad h(n) = \begin{Bmatrix} 1, 2, 2, 1 \\ \uparrow \end{Bmatrix}$$

Solution:

$x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore, the starting sample of $y(n)$ is at

$$n = n_1 + n_2 = 0 - 1 = -1$$

$x(n)$ has 4 samples, $h(n)$ has 4 samples. Therefore, $y(n)$ will have $N = 4 + 4 - 1 = 7$ samples, i.e., from $n = -1$ to $n = 5$.

Method 1 Graphical method

We know that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

From Figure 2.6, we get

$$\text{For } n = -1 \quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$$

$$\text{For } n = 0 \quad y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

$$\text{For } n = 1 \quad y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

$$\text{For } n = 2 \quad y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

$$\text{For } n = 3 \quad y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

$$\text{For } n = 4 \quad y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

$$\text{For } n = 5 \quad y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$$

$$\therefore y(n) = \begin{Bmatrix} 4, 10, 13, 13, 10, 7, 3 \\ \uparrow \end{Bmatrix}$$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 10$, $\sum_n h(n) = 6$ and $\sum_n y(n) = 60$

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved). Therefore, the result is correct.

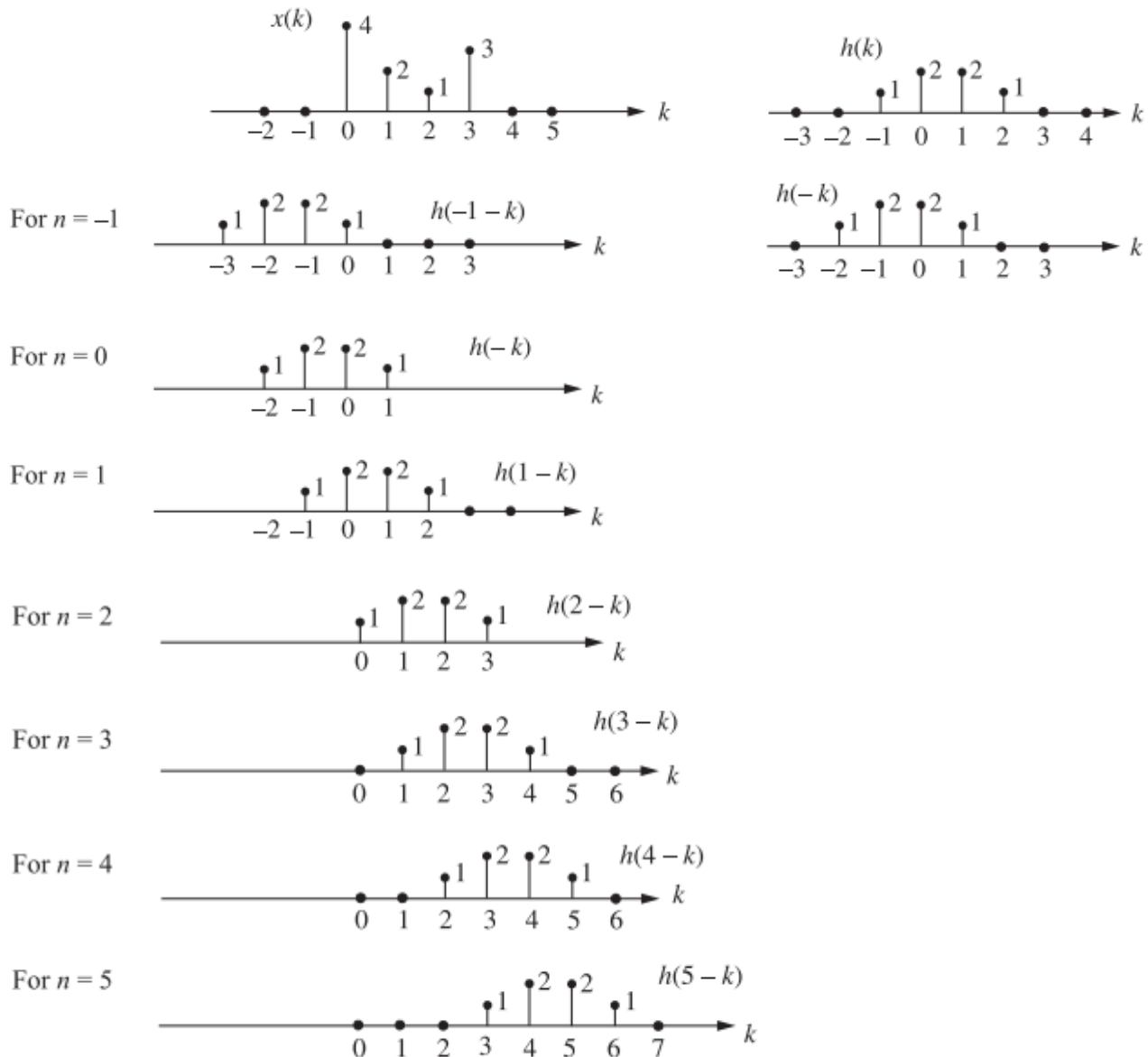


Figure 2 Operation on signals $x(n)$ and $h(n)$ to compute convolution.

Method 2 Tabular array

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 2

TABLE 2 Table for computing $y(n)$.

k		-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x(k)$		-	-	-	-	4	2	1	3	-	-	-	-
$h(-k)$		-	-	1	2	2	1	-	-	-	-	-	-
$n = -1$	$h(-1 - k)$	-	1	2	2	1	-	-	-	-	-	-	-
$n = 0$	$h(-k)$	-	-	1	2	2	1	-	-	-	-	-	-
$n = 1$	$h(1 - k)$	-	-	-	1	2	2	1	-	-	-	-	-
$n = 2$	$h(2 - k)$	-	-	-	-	1	2	2	1	-	-	-	-
$n = 3$	$h(3 - k)$	-	-	-	-	-	1	2	2	1	-	-	-
$n = 4$	$h(4 - k)$	-	-	-	-	-	-	1	2	2	1	-	-
$n = 5$	$h(5 - k)$	-	-	-	-	-	-	-	1	2	2	1	-

The starting value of $n = -1$. From the table, we can see that

$$\text{For } n = -1 \quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$$

$$\text{For } n = 0 \quad y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

$$\text{For } n = 1 \quad y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

$$\text{For } n = 2 \quad y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

$$\text{For } n = 3 \quad y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

$$\text{For } n = 4 \quad y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

$$\text{For } n = 5 \quad y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$$

$$\therefore y(n) = \left\{ \begin{array}{c} 4, 10, 13, 13, 10, 7, 3 \\ \quad \uparrow \end{array} \right\}$$

Method 3 Tabular method

Given $x(n) = \{4, 2, 1, 3\}, \quad h(n) = \begin{Bmatrix} 1, 2, 2, 1 \\ \uparrow \end{Bmatrix}$

The convolution of $x(n)$ and $h(n)$ can be computed as shown in Table 3.

TABLE 3 Table for computing $y(n)$.

		$x(n)$			
		4	2	1	3
$h(n)$	1	4	2	1	3
	2	8	4	2	6
	2	8	4	2	6
	1	4	2	1	3

$$\begin{aligned} y(n) &= 4, 8 + 2, 8 + 4 + 1, 4 + 4 + 2 + 3, 2 + 2 + 6, 1 + 6, 3 \\ &= 4, 10, 13, 13, 10, 7, 3 \end{aligned}$$

The starting value of n is equal to -1 , mark the symbol $\uparrow$ at time origin ($n = 0$).

$$\therefore y(n) = \begin{Bmatrix} 4, 10, 13, 13, 10, 7, 3 \\ \uparrow \end{Bmatrix}$$

Method 4 Matrices method

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{4, 2, 1, 3\}$

and $h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, 2, 2, 1\}$

$\uparrow$

The sequence $x(n)$ is starting at $n = 0$ and the sequence $h(n)$ is starting at $n = -1$. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $h(n)$ will start at $n = 0 + (-1) = -1$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 - 1 = 7$. Substituting the sequence values in matrix form and multiplying as shown below, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 13 \\ 13 \\ 10 \\ 7 \\ 3 \end{bmatrix}$$

$$y(n) = x(n) * h(n) = [4 \ 10 \ 13 \ 13 \ 10 \ 7 \ 3]$$

$\uparrow$

EXAMPLE 3 Find the convolution of the signals

$$x(n) = \begin{cases} 2 & n = -2, 0, 1 \\ 3 & n = -1 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-2) - \delta(n-3)$$

Solution: Given $x(n) = \begin{Bmatrix} 2, 3, 2, 2 \\ \uparrow \end{Bmatrix}$; $h(n) = \{1, -2, 3, -1\}$

$x(n)$ starts at $n_1 = -2$ and $h(n)$ starts at $n_2 = 0$. The starting sample of $y(n)$ is at $n = n_1 + n_2 = -2 + 0 = -2$. Since number of samples in $x(n)$ is 4, and the number of samples in $h(n)$ is 4, the number of samples in $y(n)$ will be $4 + 4 - 1 = 7$. So $y(n)$ exists from -2 to 4 .

Method 1 Graphical method

We know that
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

From Figure 1 , we get

For $n = -2$
$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k) = 2 \cdot 1 = 2$$

For $n = -1$
$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 2(-2) + 3 \cdot 1 = -1$$

For $n = 0$
$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 2 \cdot 3 + 3(-2) + 2 \cdot 1 = 2$$

For $n = 1$
$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2(-1) + 3(3) + 2(-2) + 2 \cdot 1 = 5$$

For $n = 2$
$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 3(-1) + 2 \cdot 3 + 2(-2) = -1$$

For $n = 3$
$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2(-1) + 2 \cdot 3 = 4$$

For $n = 4$
$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 2(-1) = -2$$

$\therefore y(n) = \begin{Bmatrix} 2, -1, 2, 5, -1, 4, -2 \\ \uparrow \end{Bmatrix}$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 9$, $\sum_n h(n) = 1$, and $\sum_n y(n) = 9$.

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved).

Therefore, the result is correct.

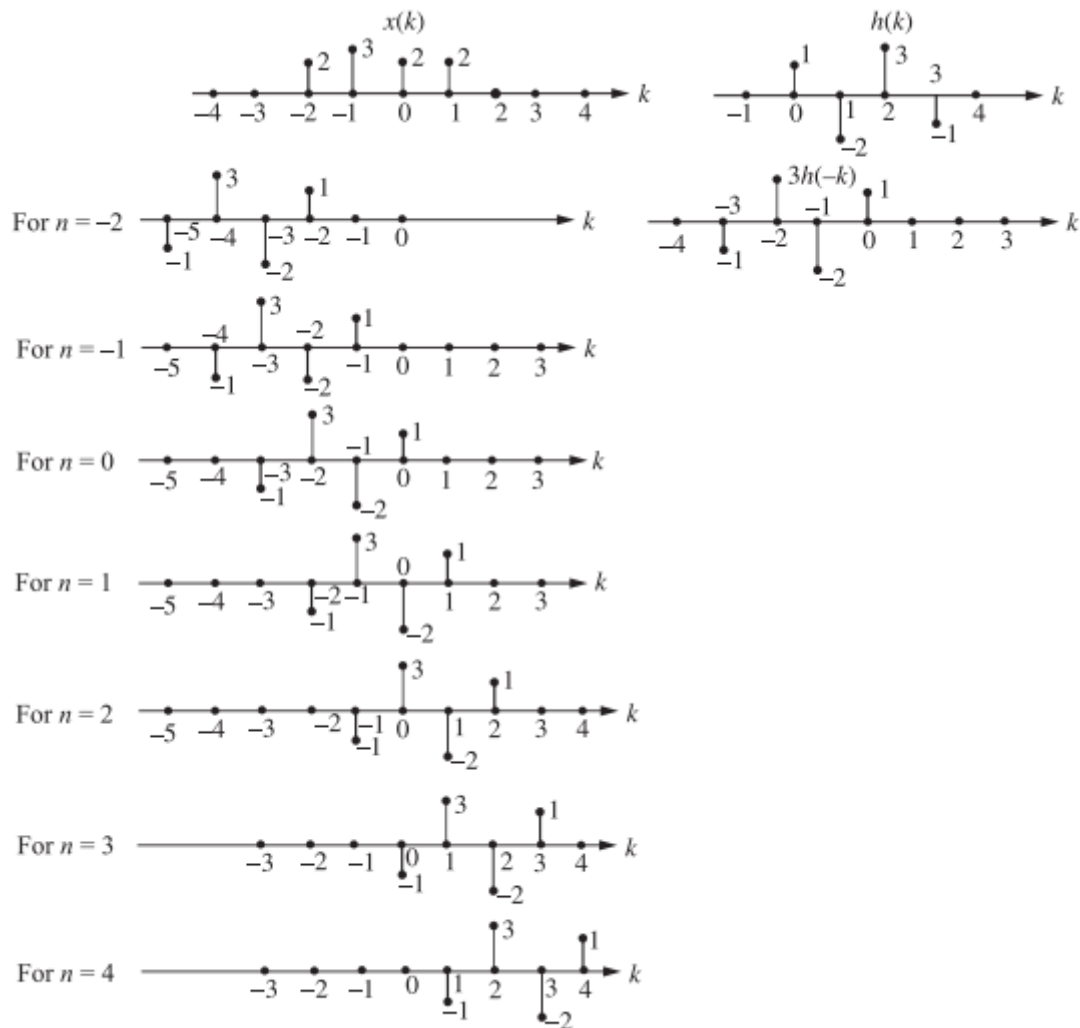


Figure 1 Operation on signals $x(n)$ and $h(n)$ to compute convolution.

Method 2 Tabular array

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 1

Table 1 Table for computing $y(n)$.

k	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$x(k)$	-	-	-	2	3	2	2	-	-	-	-	-
$h(-k)$	-	-	-1	3	-2	1	-	-	-	-	-	-
$n = -2$ $h(-2 - k)$	-1	3	-2	1	-	-	-	-	-	-	-	-
$n = -1$ $h(-1 - k)$	-	-1	3	-2	1	-	-	-	-	-	-	-
$n = 0$ $h(-k)$	-	-	-1	3	-2	1	-	-	-	-	-	-
$n = 1$ $h(1 - k)$	-	-	-	-1	3	-2	1	-	-	-	-	-
$n = 2$ $h(2 - k)$	-	-	-	-	-1	3	-2	1	-	-	-	-
$n = 3$ $h(3 - k)$	-	-	-	-	-	-1	3	-2	1	-	-	-
$n = 4$ $h(4 - k)$	-	-	-	-	-	-	-1	3	-2	1	-	-

The starting sample of $y(n)$ is at $n = -2$. $y(n)$ is calculated as shown below.
From the table, we can see that

$$\text{For } n = -2 \quad y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k) = 2 \cdot 1 = 2$$

$$\text{For } n = -1 \quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 2(-2) + 3 \cdot 1 = -1$$

$$\text{For } n = 0 \quad y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 2 \cdot 3 + 3(-2) + 2 \cdot 1 = 2$$

$$\text{For } n = 1 \quad y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2(-1) + 3(3) + 2(-2) + 2 \cdot 1 = 5$$

$$\text{For } n = 2 \quad y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 3(-1) + 2 \cdot 3 + 2(-2) = -1$$

$$\text{For } n = 3 \quad y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2(-1) + 2 \cdot 3 = 4$$

$$\text{For } n = 4 \quad y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 2(-1) = -2$$

$$\therefore y(n) = \left\{ 2, -1, 2, 5, -1, 4, -2 \right\}$$

↑

Method 3 Tabular method

$y(n) = x(n) * h(n)$ is computed by tabular method as shown in Table 2.

TABLE 2. Table for computing $y(n)$.

		$x(n)$			
		2	3	2	2
$h(n)$	1	2	3	2	2
	-2	-4	-6	-4	-4
	3	6	9	6	6
	-1	-2	-3	-2	-2

$$\therefore y(n) = 2, -4 + 3, 6 - 6 + 2, -2 + 9 - 4 + 2, -3 + 6 - 4, -2 + 6, -2$$

$$= \left\{ 2, -1, 2, 5, -1, 4, -2 \right\}$$

↑

Method 4 Matrices method

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{2, 3, 2, 2\}$
 $\uparrow$

and $h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, -2, 3, -1\}$
 $\uparrow$

The sequence $x(n)$ is starting at $n = -1$ and the sequence $h(n)$ is also starting at $n = -1$. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $y(n)$ will start at $n = -1 + (-1) = -2$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 - 1 = 7$. Substituting the sequence values in matrix form and multiplying, we get the convolution of $x(n)$ and $h(n)$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 3 & -2 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \\ -1 \\ 4 \\ -2 \end{bmatrix}$$