How can I check the matrix is a positive definite or not.

① If all eigenvalues $\lambda_i > 0 \; \forall i = 1, \dots, n$

② If all its leading principal minors are positive

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

③ If there exists a unique lower triangular matrix $L \in \mathbb{R}^{n \times n}$ with positive diagonal components s.t $A = LL^T$

Example :-

① $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite because $\lambda_1 = 2 - \sqrt{2}$, $\lambda_2 = 2 + \sqrt{2}$, $\lambda_3 = 2$

② $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is positive semi-definite

③ $\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$ is indefinite

Solution of $Ax=b$

Let $A \in R^{n\times n}$ symmetric and positive definite then the solution of $Ax=b$ is $x^* = A^{-1}b$

Definition:- Let $a, b \in R$, then the closed interval $[a,b]$ denoted the set $\{x \in R : a \le x \le b\}$. The set $\{x \in R : a < x < b\}$ represents the open interval (a,b).

Example :- $[-3,2]$ a closed interval
$(1,3)$ an open interval

Definition:- Let $x_0 \in R^n$. A norm ball of radius $r>0$ and center x_0 is given by $\{x \in R^n : \|x - x_0\| \le r\}$ and will be denoted by $B[x_0, r]$

Note: we will use $B(x_0, r)$ to denote $\{x \in R^n ; \|x - x_0\| < r\}$

Definition:- Let $x \in S \subseteq R^n$, then [illegible] called an interior point of S if there exists $r>0$ s.t. $B[x,r] \subseteq S$. The set of all points which satisfy the condition of interior point of S is called the interior of S and denoted by $int(S)$

Definition:- A set $S \subseteq R^n$ is said to be an open set if $S = int(S)$

Definition :- A set $S \in R^n$ is said to be closed if its complement in R^n; $R^n | S = \{ x \in R^n : x \notin S \}$ is open

Definition :- let $S \subset R^n$, $x \in R^n$ belongs to the closure of S, $cl(S)$ if for each $\epsilon > 0$, $S \cap B[x, \epsilon] \neq 0$ and the set S is called closed if $S = cl(S)$.

Example :- $[0,1] \cup [2,3]$ is a closed subset of R

Example :- let $S = (1,2] \cup [3,4)$, then

$cl(S) = [1,2] \cup [3,4]$ and $int(S) = (1,2) \cup (3,4)$

Definition :- The boundary of a set S is defined as

$bd(S) = cl(S) \setminus int(S)$

Definition :- A set $S \subset R^n$ is said to be bounded if there exists R s.t $(0 < R < \infty)$ and $x \in R^n$, where $S \subset B(x, R)$

Example :- $(1,2] \cup [3,10)$: a bounded set

$[0, \infty)$: not ~~a~~ bounded set

Definition :- A set S in R^n is said to be compact if it is closed and bounded

Example :- $[0,10] \cup [20,30]$ a compact set

Sequence

Definition:- A sequence $\{x^k\}$ converges to x^*, if for any given $\epsilon > 0$ there exists positive integer K s.t

$\|x^k - x^*\| \leq \epsilon \quad \forall\, k \geq K$

we write this as $x^k \to x^*$ or $\lim_{k \to \infty} x^k = x^*$

Example: ① The sequence $\{x^k\}$ where

$x^k = (1 + 2^{-k}, 1/k)^T$ converges to $(1, 0)^T$

② The sequence $\{x^k\}$ where $x^k = (-1)^k$ does not convergence.

Continous function

Definition:- Let $S \subseteq R^n$. A function $f: S \to R$ is said to be continous at $x^* \in S$ if for any given $\epsilon > 0$ there exists a $\delta > 0$ s.t $x \in S$ and $\|x - x^*\| < \delta$ implies $|f(x) - f(x^*)| < \epsilon$

Notes. ① The function f is said to be continous on $A \subset R^n$ if it is continuous at each point of A.

② when we say that f is continuous, we mean that f is continuous on its domain

③ C: class of all continuous functions

Gradient

Definition: A continuous function, $f: R^n \to R$ is said to be continuously differentiable at $x \in R^n$, if $\frac{\partial f}{\partial x_i}(x)$ exists and is continuous, $i = 1, 2, \cdots, n$

Note: C^1: class of functions whose first partial derivatives are continuous

Definition: we define the gradient of f at x to be the vector

$$g(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)^T$$

Directional Derivatives

Definition: Let $S \subset R^n$ be an open set and $f: S \to R$, f continuously differentiable in S, then for any $x \in S$ and any nonzero $d \in R^n$, the directional derivative of f at x in the direction d, defined by

$$\frac{\partial f}{\partial d}(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon d) - f(x)}{\epsilon} \text{ exists}$$

and equals $\nabla f(x)^T d$

Example: $\phi: R \to R$ as $\phi(t) = f(x + td)$

$$\frac{d\phi}{dt}(\alpha) = \nabla f(x + \alpha d)^T d$$

when $\alpha = 0 \Rightarrow \frac{\partial f}{\partial d}(x) = \nabla f(x)^T d$

Definition:- A continuously differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be twice continuously differentiable at $x \in \mathbb{R}^n$, if $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ exists and is continuous.

C^2: class of twice continuously differentiable functions

Definition:- Let $f \in C^2$, we define the Hessian of f at x to be the matrix

$$H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Definition:- Let $S \subseteq \mathbb{R}^n$ be an open set and $f: S \to \mathbb{R}$, f twice continuously differentiable in S. Then, for any $x \in S$ and any nonzero $d \in \mathbb{R}^n$, the second directional derivative of f at x in the direction of d equals $d^T \nabla^2 f(x) d$.

Example:- consider the function $f(x_1, x_2) = x_1 e^{(-x_1^2 - x_2^2)}$

The gradient of f at $x = (x_1, x_2)^T$ is

$$g(x) = \nabla f(x) = \begin{bmatrix} (1 - 2x_1^2) e^{(-x_1^2 - x_2^2)} \\ -2x_1 x_2 e^{(-x_1^2 - x_2^2)} \end{bmatrix}$$

The Hessian of f at $x = (x_1, x_2)^T$ is

$$H = \nabla^2 f(x) = \begin{bmatrix} (4x_1^3 - 6x_1) e^{(-x_1^2 - x_2^2)} & -2x_2(1 - 2x_1^2) e^{(-x_1^2 - x_2^2)} \\ -2x_2(1 - 2x_1^2) e^{(-x_1^2 - x_2^2)} & (4x_1 x_2^2 - 2x_1) e^{(-x_1^2 - x_2^2)} \end{bmatrix}$$