$f''(1) = 0$ and $f''(-1) = 0$ so we will find the third derivative of f,

$f'''(x) = 120x^3 - 72x$

$\Rightarrow f'''(1) = 48 > 0$
$\Rightarrow f'''(-1) = -48 < 0$
$\Rightarrow x = 1$ and $x = -1$ are saddle points

Note:- The standard method for find the stationary point of the function by solving the nonlinear equation

$$g(x) = f'(x) = 0$$

by finding the roots or real roots of $g(x)$, but it may not always easy, for example if we have the problem to minimize $f(x) = x^2 + e^x$ so we must solve the equation $g(x) = 2x + e^x = 0$ so we need an algorithm to find x which satisf $g(x) = 0$,

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The methods of numerical optimization are divided to two types

① Derivative-free methods (search methods)

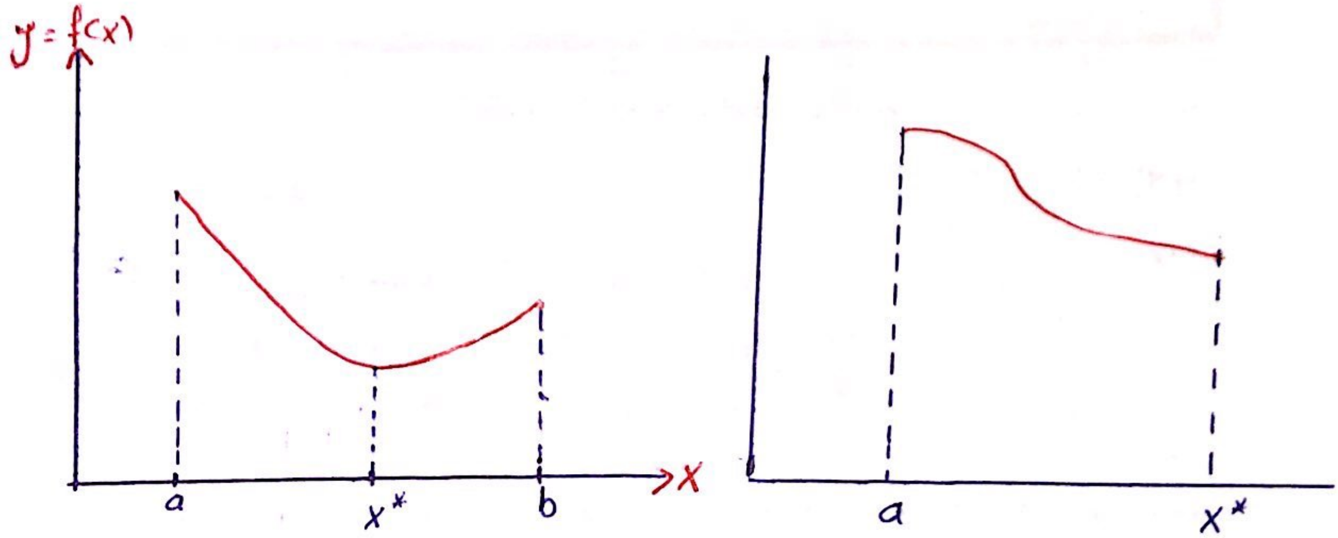
② Derivative-based methods (Approximation methods)

Let $\phi : R \longrightarrow R$ and let $\min_{x \in R} \phi(x)$ be the problem, and let x^* be the minimum point of $\phi(x)$ and $x^* \in [a,b]$.

Definition :- The function $\phi : R \rightarrow R$ is said to be unimodal on $[a,b]$ if f has a minimum $x^* \in [a,b]$ such that if for $a < x_1 < x_2 < b$

$$\text{if } \quad x_2 < x^* \Rightarrow \phi(x_1) > \phi(x_2)$$

$$\text{if } \quad x_1 > x^* \Rightarrow \phi(x_2) > \phi(x_1)$$



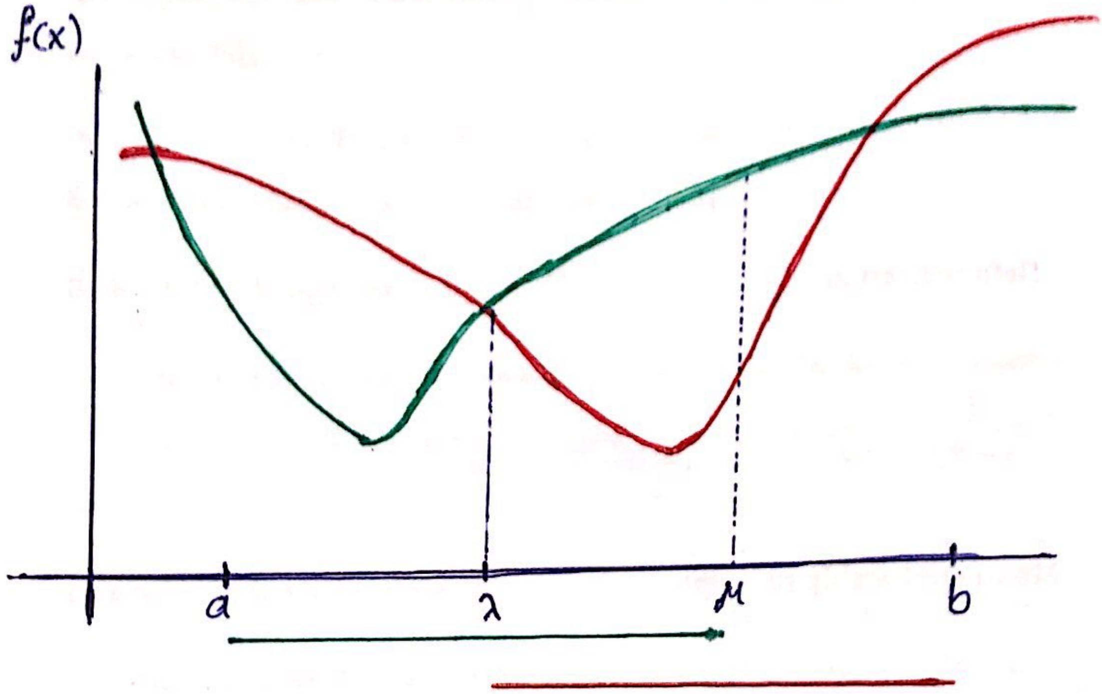
Unimodal functions

There are three different methods of derivatives-free methods (search method):

1. Dichotomous search
2. Fibonacci search
3. Golden-section search

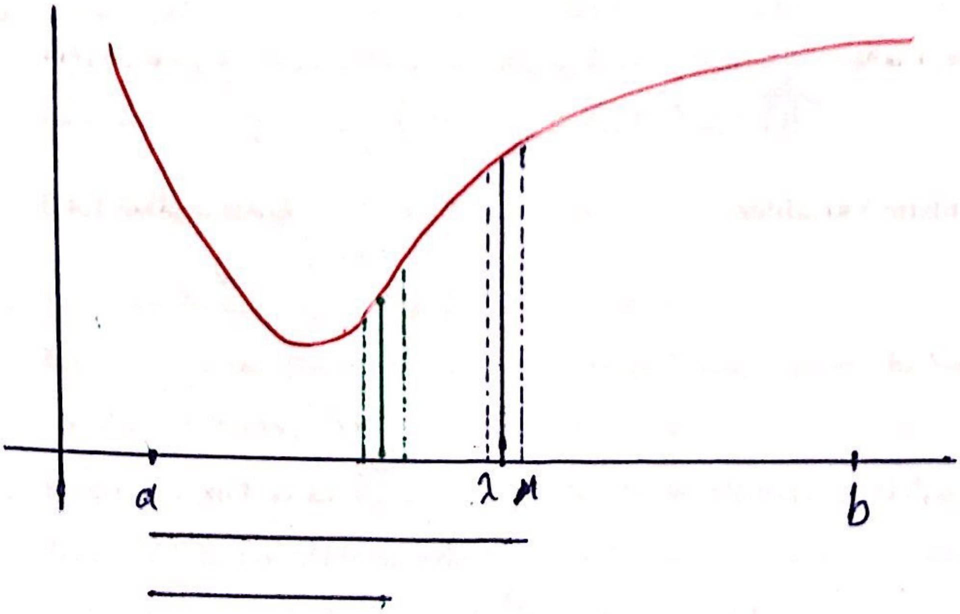
The requirement of derivative-free methods (search method) is that f unimodal function on the interval [a, b]

① Dichotomous search method



If we see the above functions (green function and red function) then we note that function is unimodal function on the interval [a, b], so each function has one minimum in the interval [a, b] and the green function has a minimum in the interval [a, λ] and the red function has a minimum in the interval [λ, b], so the three points a, λ, b it's not enough to determine the minimum of each function, so we need another point μ as shown in the above figure, after that we can determine the subinterval which contain the minimum.

Now, If we have the following unimodal function



we place λ and μ symmetrically, each at a distance ϵ from mid-point of $[a,b]$

$$\Rightarrow \quad \mu = \frac{a+b}{2} + \epsilon \quad \& \quad \lambda = \frac{a+b}{2} - \epsilon$$

So we have two intervals $[a,\mu]$ and $[\lambda, b]$ and we can determine the interval which contain the interval

<u>The Algorithm of Dichotomous search method</u>

- input: initial interval of uncertainty, $[a,b]$
- initialization: $k=0$, $a^k = a$, $b^k = b$, $\epsilon > 0$, l (accuracy solution)
- while $(b^k - a^k) > l$
- $\lambda^k = \frac{a^k+b^k}{2} - \epsilon$ & $\mu^k = \frac{a^k+b^k}{2} + \epsilon$
- if $f(\lambda^k) \geq f(\mu^k)$
- $a^{k+1} = \lambda^k$ & $b^{k+1} = b^k$
- else
- $b^{k+1} = \mu^k$ & $a^{k+1} = a^k$
- endif
- $k := k+1$
- endwhile
- output: $x^* = \frac{a^k+b^k}{2}$

Examples:- $\min_{x} (1/4)x^4 - (5/3)x^3 - 6x^2 + 19x - 7$

in the interval $[-4, 0]$, by use Dichotomous search method

Solution:- $f(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3 - 6x^2 + 19x - 7$

$a^0 = -4$, $b^0 = 0$, $\epsilon = 0.02$, $l = 0.001$

$|a^0 - b^0| = |-4-0| = 4 > l$

$$\mu^0 = \frac{a^0 + b^0}{2} - \epsilon = \frac{-4+0}{2} + 0.02 = -1.98$$

$$\lambda^0 = \frac{a^0 + b^0}{2} + \epsilon = \frac{-4+0}{2} - 0.02 = -2.02$$

$f(\mu^0) = -51.3627$, $f(\lambda^0) = -51.9626$

$\Rightarrow f(\mu^0) > f(\lambda^0)$

So $b^1 = \mu^0 = -1.98$ & $a^1 = a^0 = -4$

$\Rightarrow [a^1, b^1] = [-4, -1.98]$, $|a^1 - b^1| = 2.02$

Now, $$\mu^1 = \frac{a^1 + b^1}{2} + \epsilon = \frac{-4-1.98}{2} + 0.02 = -2.97$$

$$\lambda^1 = \frac{a^1 + b^1}{2} - \epsilon = \frac{-4-1.98}{2} - 0.02 = -3.01$$

$f(\mu^1) = -53.2399$ & $f(\lambda^1) = -52.5777$

$\Rightarrow f(\mu^1) < f(\lambda^1)$

So $a^2 = \lambda^1 = -3.01$ & $b^2 = b^1 = -1.98$

$\Rightarrow [a^2, b^2] = [-3.01, -1.98] \Rightarrow |a^2 - b^2| = 1.03$

and so on $\longrightarrow$