Sufficient optimility Conditions

① Let $f: R \to R$, $f \in C^{\infty}$

② Let us assume that f is not a constant functions.

③ Let the k-th derivative of f at x be denoted by $f^{(k)}(x)$

④ Consider the problem $\min_{x \in R} f(x)$

* **Result** :- x^* is a local minimum if and only if the first non-zero element of the sequence $\{f^{(k)}(x)\}$ is positive and occurs at even positive k.

* **Result** :- consider the problem $\max_{x \in R} f(x)$, x^* is a local maximum if and only if the first non-zero element of the sequence $\{f^{(k)}(x)\}$ is negative and occurs at positive k.

Example :- consider we have the following problem

$$f(x) = (x^2 - 1)^2$$

To find the stationary points of $f(x) = (x^2-1)^2$

$$\Rightarrow f'(x) = 0 \Rightarrow 2(x^2-1)(2x) = 0 \Rightarrow 4x(x^2-1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1$$

$f''(1) = f''(-1) = 8 > 0 \Rightarrow x = 1$ and $x = -1$ are strict local minimum

$f''(0) = -4 < 0 \Rightarrow x = 0$ is a strict local maximum

Example :- Let we have the following problem

$\min\limits_{x \in \mathbb{R}} (x-2)^2$, So we will find the stationary points

$f'(x) = 2(x-2) \Rightarrow x^* = 2$

$\Rightarrow f''(2) = 2 > 0 \Rightarrow x^* = 2$ is a strict local minimum

Example : Let we have the following problem

$\min\limits_{x \in \mathbb{R}} f(x) \Rightarrow \min\limits_{x \in \mathbb{R}} x^4$

$\Rightarrow f'(x) = 4x^3 \Rightarrow f'(x) = 0 \Rightarrow x = 0$

$f''(x) = 12x^2 \Rightarrow f''(x) = 0$

$f'''(x) = 24x \Rightarrow f'''(x) = 0$

$f''''(x) = f^{(4)}(x) = 24 \Rightarrow f''''(x) > 0$

$\Rightarrow x^* = 0$ is a strict local minimum

Example :- Assume the following problem

$\min\limits_{x \in \mathbb{R}} (x^2-1)^3 \Rightarrow f(x) = (x^2-1)^3$

$\Rightarrow f'(x) = 3(x^2-1)^2(2x) \Rightarrow f'(x) = 6x(x^2-1)^2$

$\Rightarrow f'(x) = 0 \Rightarrow 6x(x^2-1)^2 = 0 \Rightarrow x = 0, x = 1, x = -1$

$f''(x) = 30x^4 - 36x^2 + 6 \Rightarrow f''(0) = 6 > 0$

$\Rightarrow x^* = 0$ is a strict local minimum

$f''(1) = 0$ and $f''(-1) = 0$ so we will find the third derivative of f,

$f'''(x) = 120x^3 - 72x$

$\Rightarrow f'''(1) = 48 > 0$
$\Rightarrow f'''(-1) = -48 < 0$ $\Big\}$ $\Rightarrow x = 1$ and $x = -1$ are saddle points

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Note:- The standard method for find the stationary points of the function by solving the nonlinear equation

$$g(x) = f'(x) = 0$$

By finding the roots or real roots of $g(x)$, but it may not always easy, for example if we have the problem to minimize $f(x) = x^2 + e^x$ so we must solve the equation $g(x) = 2x + e^x = 0$ so we need an algorithm to find x which satisf $g(x) = 0$,