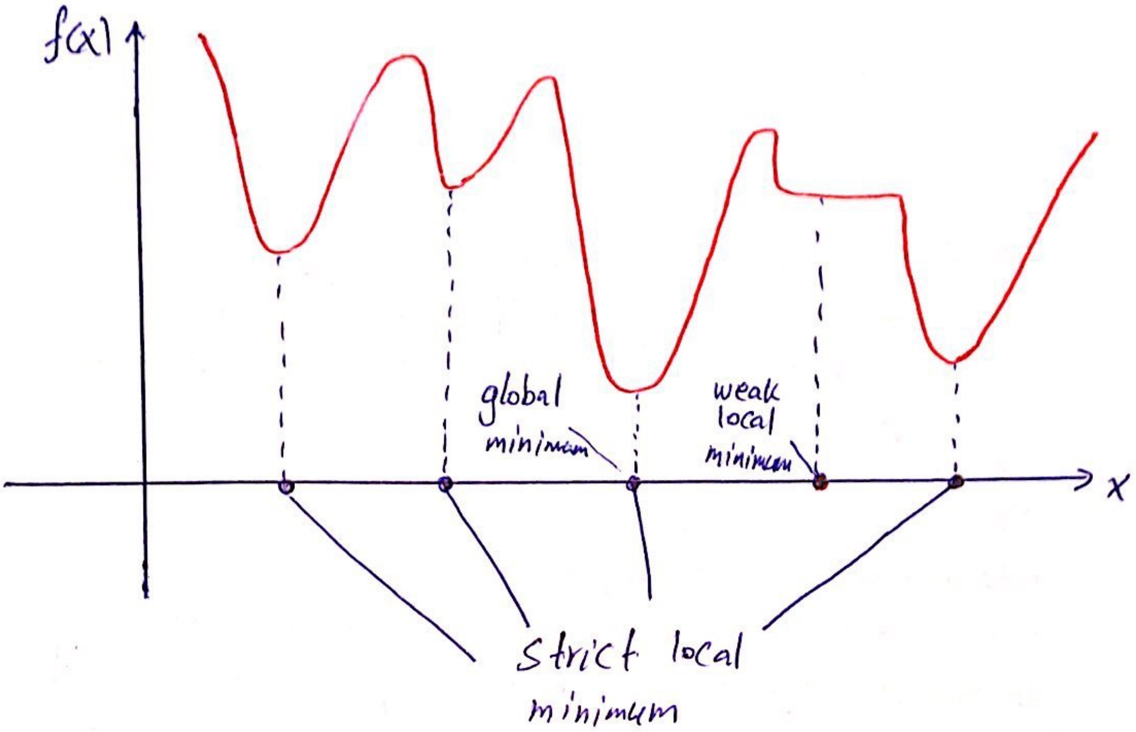
Definition:- Let $f: X \to R$ be a function, then x^* is said to be a strict local minimum of f if $f(x^*) < f(x)$ $\forall\ x \in X \cap B(x^*, \delta)$, $x \neq x^*$, and it is called weak local minimum if $f(x^*) \leq f(x)$ $\forall\ x \in X \cap B(x^*, \delta)$, $x \neq x^*$.



Notes:- ① Every global minimum is also a local minimum

② It may not be possible to identify a global minimum by finding all local minimum.

unconstrained one-dimensional optimization problem

$$\min_{x \in R} f(x)$$

where $f: R \longrightarrow R$

The Necessary and Sufficient conditions for a local minimum

1. Necessary conditions :- Conditions satisfied by every local minimum.
2. Sufficient conditions :- Conditions which guarantee a local minimum.

First order Necessary Condition

Let $f: R \longrightarrow R$, $f \in C'$ and consider the problem

is $\min_{x \in R} f(x)$.

* If x^* is a local minimum of f, then $f'(x^*) = 0$

proof: Let $f'(x^*) > 0$ and let $f \in C'$ and let $D = (x^* - \delta, x^* + \delta)$ be chosen such that $f'(x) > 0$ $\forall x \in D$, therefore for any $x \in D$, using first order truncated Taylor Series

$$f(x) = f(x^*) + f'(\bar{x})(x - x^*) \qquad \text{where } \bar{x} \in (x^*, x)$$

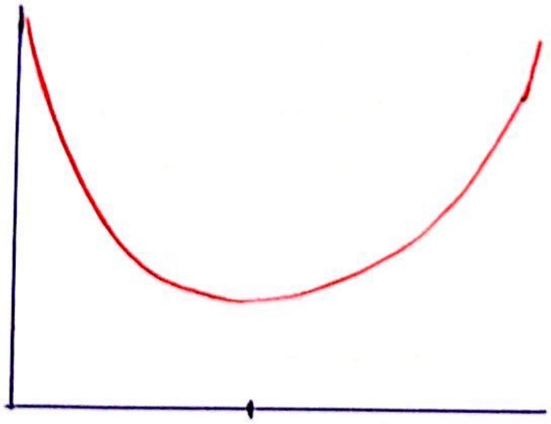
Now, we choose $x \in (x^* - \delta, x^*)$ we get

$$f(x) < f(x^*), \text{ contradication}$$

Similarity we can show that $f(x) < f(x^*)$ when we let $f'(x^*) < 0$ (Homework)

Example: In the following two examples we will show the first order necessary condition of local minimum

①

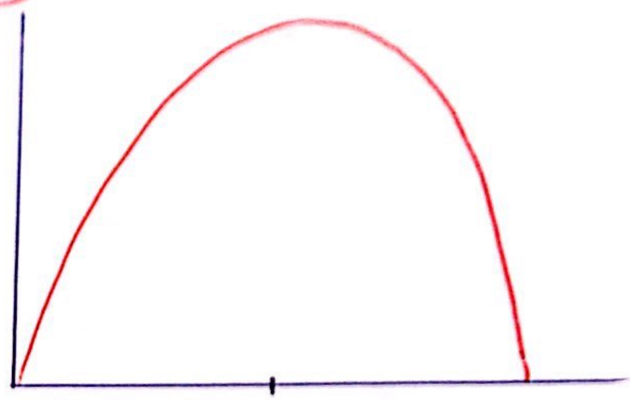


$f(x) = (x-2)^2$

$f'(2) = 0$

$f''(2) = 4 > 0$

②

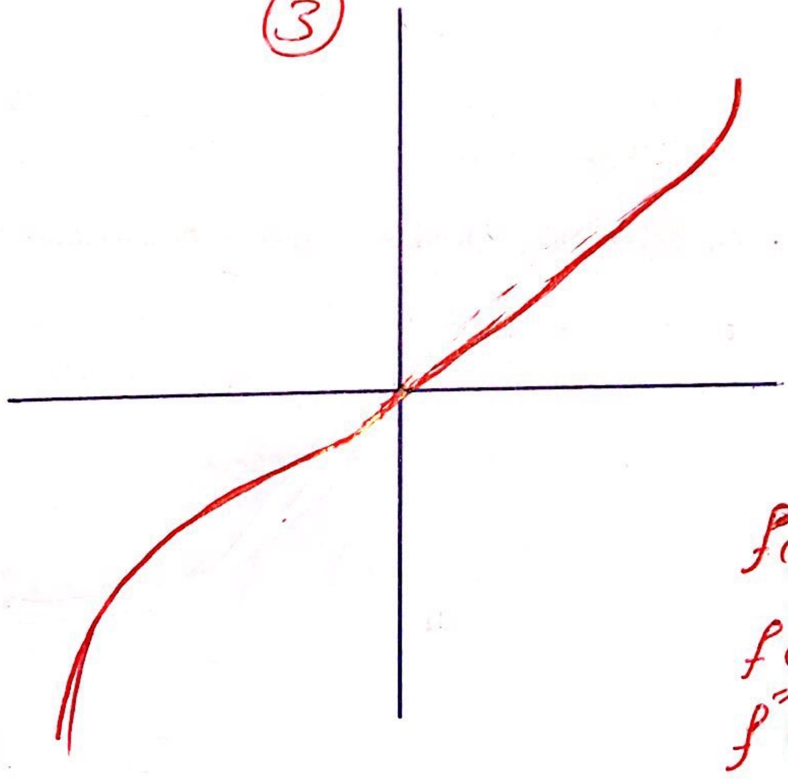


$f(x) = -x^2$

$f'(0) = 0$

$f''(0) = -2$

③



$f(x) = x^3$

$f'(0) = 0$

$f''(0) = 0$

Definition: let f be a function from R to R, then $x^* \in R$ is said to be stationary point if $f'(x^*)=0$

Note:- $f'(x^*)=0$ is a necessary condition but not sufficient condition for local minimum

* Now the quastion is, how do we ensure that the stationary point is a local minimum

Second order Necessary Conditions

If x^* is a local minimum of f, then $f'(x^*)=0$ and $f''(x^*)\geq 0$

proof: By the first order necessary conditions, $f'(x^*)=0$

Suppose $f''(x^*)<0$, let $f\in C^2$

Let $D=(x^*-\delta, x^*+\delta)$ be chosen such that $f''(x)<0$ $\forall\ x\in D$. Therefore, for any $x\in D$, using the second order truncated Taylor Series

$$f(x)=f(x^*)+f'(x)(x-x^*)+\frac{1}{2}f''(\bar{x})(x-x^*)^2$$

where $\bar{x}\in(x^*,x)$, using $f'(x^*)=0$ and $f''(\bar{x})<0$ $\forall\ x\in D$, we get

$f(x)<f(x^*)$ Contradiction

Note: The second order necessary condition also is not sufficient, we can check that by example ③ because of $x^* = 0$, $f'(x^*) = 0$, $f''(x^*) = 0$, but x^* is a saddle point

Second order Sufficient conditions

If $x^* \in \mathbb{R}$ such that $f'(x^*) = 0$ and $f''(x^*) > 0$, then x^* is a strict local minimum of f over $\mathbb{R}$.

proof:- let $f \in C^2$, and let $D = (x^* - \delta, x^* + \delta)$ be chosen such that $f''(x) > 0 \; \forall \, x \in D$. Therefore, for any $x \in D$, using second order truncated Taylor Series

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} f''(\bar{x})(x - x^*)^2$$

where $\bar{x} \in (x^*, x)$

Then $f'(x^*) = 0 \Rightarrow f(x) - f(x^*) = \frac{1}{2} f''(\bar{x})(x - x^*)^2 > 0$

That is $f(x) > f(x^*) \; \forall \, x \in D$

$\Rightarrow x^*$ is a strict local minimum

Note:- The second order sufficient conditions

① guarantee that the local minimum is strict

② Not necessary ($f(x) = x^4$, $x^* = 0$ is strict local minimum, but $f'(x^*) = f''(x^*) = 0$)