Tylor Series

C^∞: class of all function for which the derivatives of any order is continuous

Let $f: R \to R$ and let $f \in C^\infty$

Let x^0 be the point about which we write the Taylor series

$$f(x) = f(x^0) + f'(x_0)(x - x^0) + \frac{1}{2} f''(x^0)(x - x^0)^2 + \cdots$$

If we use only $f'(x)$, then $f(x)$ a x^0 can be approximated by

$$f(x) \simeq f(x^0) + f'(x^0)(x - x_0)$$

Similarity, using $f'(x^0)$ and $f''(x^0)$, then the quadratic approximation of f at x^0 is

$$f(x) \simeq f(x^0) + f'(x^0)(x - x^0) + \frac{1}{2} f''(x^0)(x - x^0)$$

Truncated Tylor Series (First order)

Let $f: R^n \to R$, $f \in C^1$, $x^0 \in R^n$

Then for every $x \in R^n$

$$f(x) = f(x^0) + \nabla f(\bar{x})(x - x^0)$$

where $\bar{x}$ is some point that lies on the line segment joining x and x^0: $\bar{x}$ depends on x, x^0 and f

(21)

Truncated Taylor Series (second order)

Let $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C^2$, $x^0 \in \mathbb{R}^n$

Then for every $x \in \mathbb{R}^n$

$$f(x) = f(x^0) + \nabla f(x^0)(x - x^0) + \frac{1}{2}(x - x^0)^T \nabla^2 f(\bar{x})(x - x^0)$$

where $\bar{x}$ is some point that lies on the line segment joining x and x^0: $\bar{x}$ depends on x, x^0 and f.

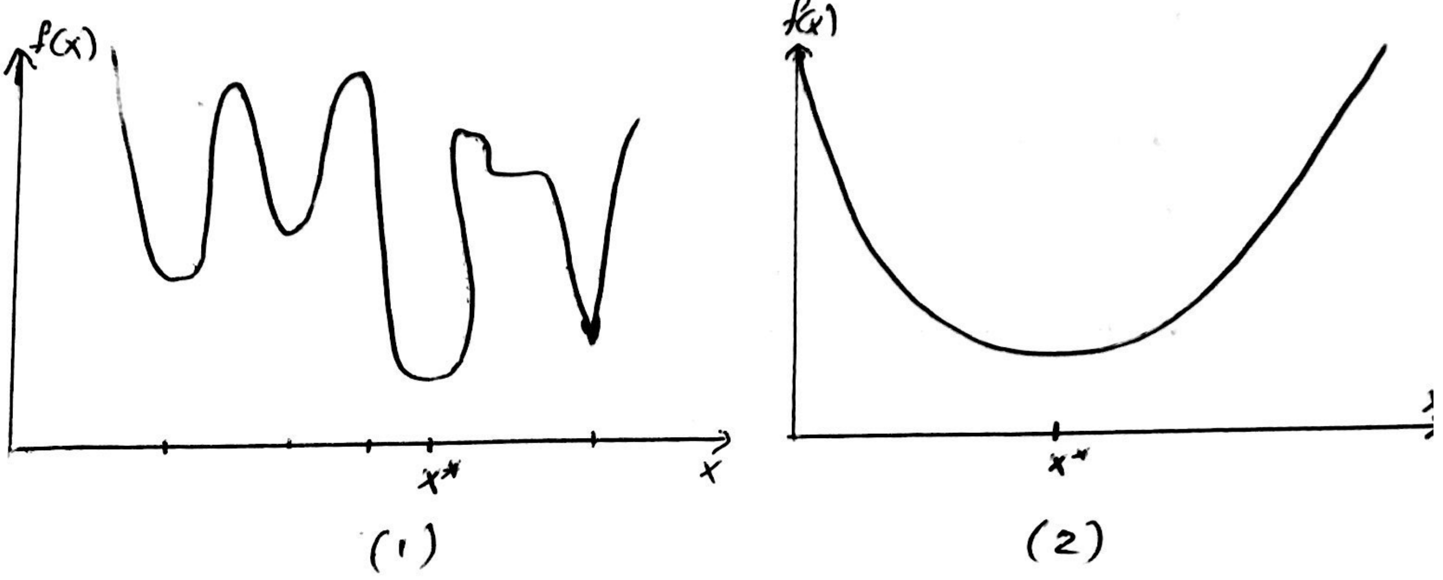
Unconstrained Optimization

Let $X \in R^n$ and $f : X \to R$
consider the problem (constrained optimization)

$$\min_{x} f(x)$$
$$\text{s.t } x \in X$$

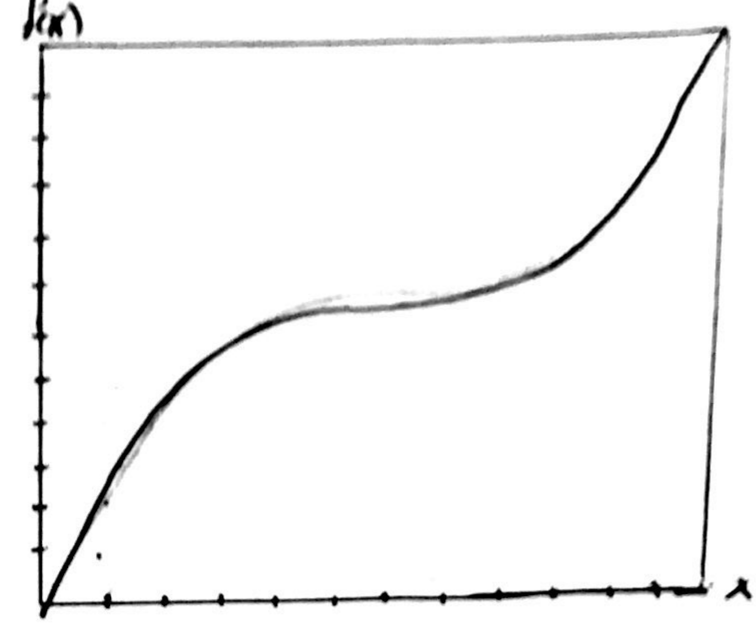
unconstrained optimization problem is a special case of constrained optimization problem.

Definition; Let $f: X \to R$ be a function then $x^* \in X$ is said to be a global minimum of f over X if $f(x^*) \leq f(x) \quad \forall x \in X$.



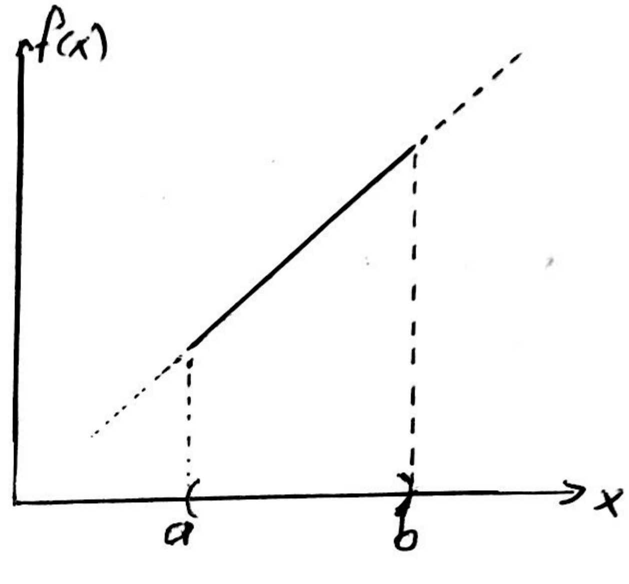
Definition: Let $f: X \to R$ be a function then $x^* \in X$ is said to be a local minimum of f if $f(x^*) \leq f(x)$ $\forall\ x \in X \cap B(x^*, \delta),\ x \neq x^*$

Example(1):- Let $X = \mathbb{R}$, $f: X \to \mathbb{R}$ defined as $f(x) = x^3$



f does't have neither a minmum nor a maximum

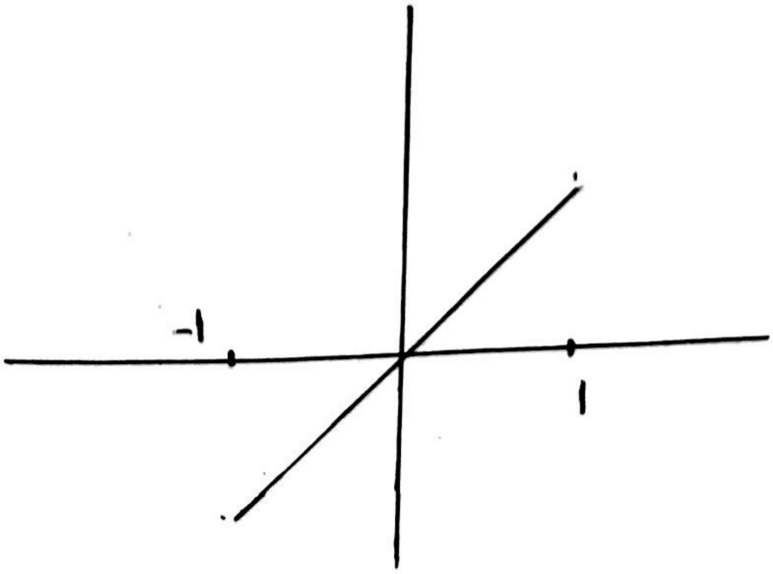
Example(2):- Let $X = (a,b)$, $f: X \to \mathbb{R}$ defined as
$f(x) = x$



f doesn't have neither a minimum nor a maximum

Example(3):- $X = [-1, 1]$, $f: X \to R$ defined as

$$f(x) = x \quad \text{if } -1 < x < 1$$
$$f(x) = 0 \quad \text{otherwise}$$



f doesn't have a minimum nor a maximum

Now, we discuss the previous three examples

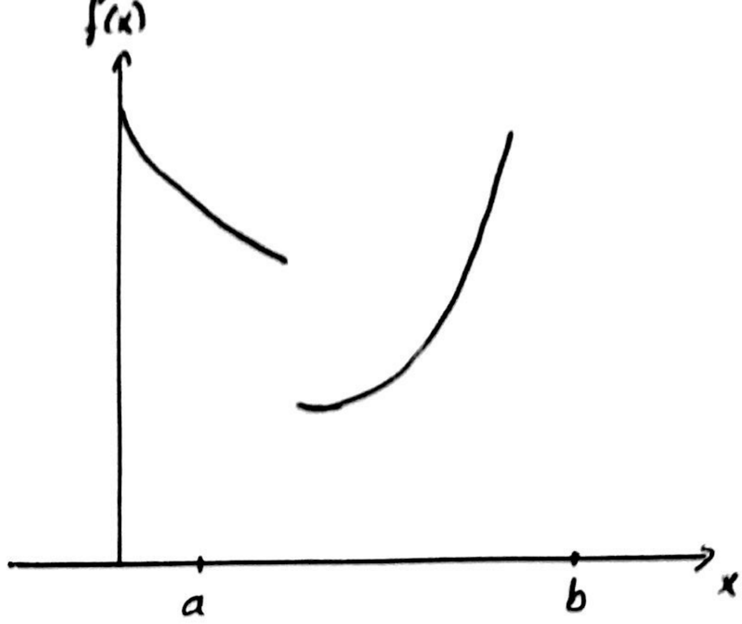
* For the fist example :- the domain X is not bounded but it is closed, So X is not compact set.

* For the second example :- the domain X is bounded but it is not closed, So X is not compact set.

* For the third example :- the domain X is bounded and closed set, So X is compact set

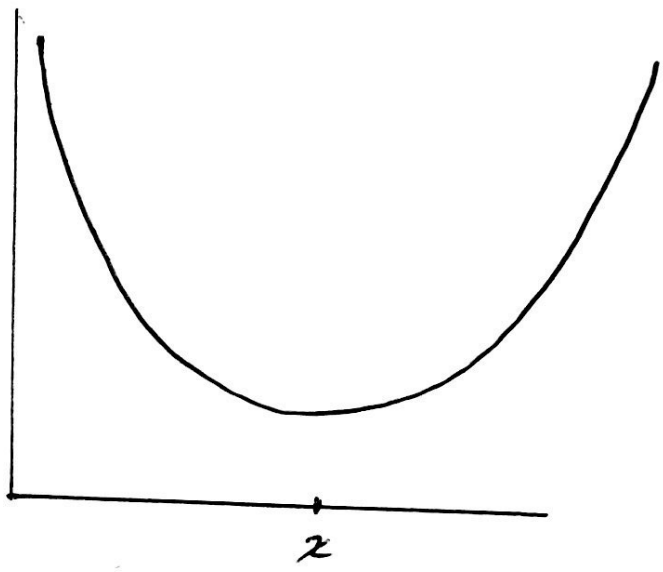
Theorem:- Let $X \subset R^n$ be a nonempty compact set and $f: X \to R$ be a continuous function on X. Then, f attains a maximum and a minimum on X; that is x_1 and x_2 in X such that $f(x_1) \geq f(x) \geq f(x_2) \quad \forall\, x \in X$

Example :- Let $X = [a, b]$ and f defined as follow



The function f is not continuous, but f attains a minimum. So the previous theorem provides only sufficient dition for the existence of optimal point.

Example :- Let $X = R$, $f : X \to R$ defined as $f(x) = (x-2)^2$



The function f has a not compact domain, but f attains a minimum