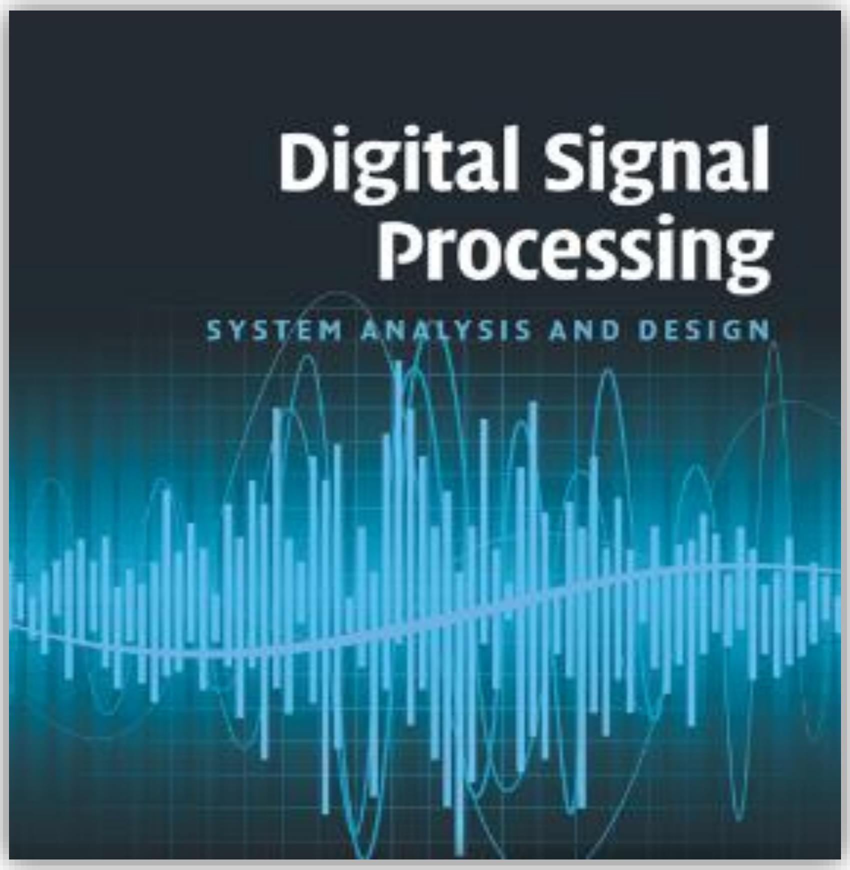


# Tikrit University

## Computer Science Dept.

## Third Class

## Lecture 7

Asst.Prof.Dr.Eng.Zaidoon.T.AL-Qaysi

## Convolution

Convolution is a mathematical way of combining two signals to form a third signal. It is the single most important technique in Digital Signal Processing. Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

A discrete-time system performs an operation on an input signal based on a predefined criterion to produce a modified output signal. The input signal $x(n)$ is the system excitation and the output signal $y(n)$ is the system response. This transform operation is shown in Figure 1.

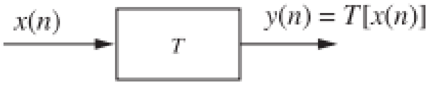


**Figure 1** A discrete-time system.

If the input to the system is a unit impulse, i.e. $x(n) = \delta(n)$, then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)]$$

The system is assumed to be initially relaxed, i.e. the system has zero initial conditions.

We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

### Example 1

Consider the special case of a finite duration sequence given as

$x(n) = \left\{-2,\ \underset{\uparrow}{4},\ 0,\ 3\right\}$ Resolve the sequence $x(n)$ into a sum of weighted impulse response

### Solution

Since the sequence $x(n)$ is non-zero for the time instants $n = -1, 0, 1, 2$, we need impulses at delay $k = -1, 0, 1, 2$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$

Using the above equation, we get

$$x(n) = 2\delta(n+1) + 4\delta(n) + 0\delta(n-1) + 3\delta(n-2).$$

So the system response $y(n)$ is given by

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

For a linear system, the above equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$$

The response due to shifted impulse sequence $\delta(n-k)$ can be denoted by $h(n, k)$, i.e.

$$h(n, k) = T[\delta(n-k)]$$

For a shift-invariant system,

Delayed output = Output due to delayed input

i.e. $$h(n-k) = h(n, k)$$

Therefore, $$T[\delta(n-k)] = h(n-k)$$

Therefore, the equation for $y(n)$ reduces to

Therefore, the equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

So we can conclude that:

For a linear time invariant system, if the input sequence $x(n)$ and the impulse response $h(n)$ are given, the output sequence $y(n)$ can be found using the equation:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

This is known as convolution sum and is represented as

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

where $*$ denotes the convolution operation.

This is an extremely powerful and useful result that allows us to compute the system output for any input signal excitation.

### *Properties of convolution*

1. Commutative property: $x(n) * h(n) = h(n) * x(n)$
2. Associative property: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3. Distributive property: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
4. Shifting property:
   If $x(n) * h(n) = y(n)$, then $x(n-k) * h(n-m) = y(n-k-m)$
5. Convolution with an impulse: $x(n) * \delta(n) = x(n)$

## CONVOLUTION OF FINITE SEQUENCES

In practice, we often deal with sequences of finite length, and their convolution may be found by several methods. The convolution $y(n)$ of two finite-length sequences $x(n)$ and $h(n)$ is also of finite length and is subject to the following rules, which serve as useful consistency checks:

1. The starting index of $y(n)$ equals the sum of the starting indices of $x(n)$ and $h(n)$.
2. The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ and $h(n)$.
3. The length $L_y$ of $y(n)$ is related to the lengths $L_x$ and $L_h$ of $x(n)$ and $h(n)$ by $L_y = L_x + L_h - 1$.

### Method 1 Linear Convolution Using Graphical Method

*Step* 1: Choose the starting time $n$ for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$ is a good choice.
*Step* 2: Express both the sequences $x(n)$ and $h(n)$ in terms of the index $k$.
*Step* 3: Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by $n$ to the right if $n$ is positive and to the left if $n$ is negative to obtain $h(n-k)$.
*Step* 4: Multiply the two sequences $x(k)$ and $h(n-k)$ element by element and sum the products to get $y(n)$.
*Step* 5: Increment the index $n$, shift the sequence $h(n-k)$ to the right by one sample and perform Step 4.
*Step* 6: Repeat Step 5 until the sum of products is zero for all remaining values of $n$.

### Method 2 Linear Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given $N$ sample sequences. Let $x_3(n)$ be the $N$ sample sequence obtained by linear convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

*Step* 1: Change the index from $n$ to $k$, and write $x_1(k)$ and $x_2(k)$.
*Step* 2: Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.
*Step* 3: Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.
*Step* 4: Shift the sequence $x_2(-k)$, $q$ times to get the sequence $x_2(q-k)$. If $q$ is positive, then shift the sequence to the right and if $q$ is negative, then shift the sequence to the left.
*Step* 5: The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k)\,x_2(q-k)$$

Determine the product sequence $x_1(k)x_2(q-k)$ for one period.
*Step* 6: The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of $n$ to get the sequence $x_3(n)$.

## Method 3 Linear Convolution Using Tabular Method

Given $x(n) = \{x_1, x_2, x_3, x_4\}$, $h(n) = \{h_1, h_2, h_3, h_4\}$

The convolution of $x(n)$ and $h(n)$ can be computed as per the following procedure.

*Step* 1: Write down the sequences $x(n)$ and $h(n)$ as shown in Table 2.1.
*Step* 2: Multiply each and every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.
*Step* 3: Group the elements in the table by drawing diagonal lines as shown in table.
*Step* 4: Starting from the left sum all the elements in each strip and write down in the same order.

$$y(n) = x_1h_1, x_1h_2 + x_2h_1, x_1h_3 + x_2h_2 + x_3h_1, x_1h_4 + x_2h_3 + x_3h_2 + x_4h_1, x_2h_4 + x_3h_3 + x_4h_2, x_3h_4 + x_4h_3, x_4h_4$$

*Step* 5: Mark the symbol ↑ at time origin ($n = 0$).

**TABLE 1** Table for Computing $y(n)$

| $h(n)$ \ $x(n)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $h_1$ | $x_1h_1$ | $x_2h_1$ | $x_3h_1$ | $x_4h_1$ |
| $h_2$ | $x_1h_2$ | $x_2h_2$ | $x_3h_2$ | $x_4h_2$ |
| $h_3$ | $x_1h_3$ | $x_2h_3$ | $x_3h_3$ | $x_4h_3$ |
| $h_4$ | $x_1h_4$ | $x_2h_4$ | $x_3h_4$ | $x_4h_4$ |

## Method 4 Linear Convolution Using Matrices

If the number of elements in $x(n)$ are $N_1$ and in $h(n)$ are $N_2$, then to find the convolution of $x(n)$ and $h(n)$ form the following matrices:

1. Matrix $H$ of order $(N_1 + N_2 - 1) \times N_1$ with the elements of $h(n)$
2. A column matrix $X$ of order $(N_1 \times 1)$ with the elements of $x(n)$
3. Multiply the matrices $H$ and $X$ to get a column matrix $Y$ of order $(N_1 + N_2 - 1)$ that has the elements of $y(n)$, the convolution of $x(n)$ and $h(n)$.

$$\underbrace{\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ h(N_2-1) & h(N_2-2) & \cdots & h(0) \\ 0 & h(N_2-1) & \cdots & h(1) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & h(N_2-1) \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x(N_1-1) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y(N_1+N_2-1) \end{bmatrix}}_{Y}$$

**EXAMPLE 2** Determine the convolution sum of two sequences:

$$x(n) = \{4, 2, 1, 3\}, \qquad h(n) = \left\{\begin{matrix} 1, 2, 2, 1 \\ \uparrow \end{matrix}\right\}$$

***Solution:***

$x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore, the starting sample of $y(n)$ is at

$$n = n_1 + n_2 = 0 - 1 = -1$$

$x(n)$ has 4 samples, $h(n)$ has 4 samples. Therefore, $y(n)$ will have $N = 4 + 4 - 1 = 7$ samples, i.e., from $n = -1$ to $n = 5$.

### *Method 1 Graphical method*

We know that $$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

From Figure 2.6, we get

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

For $n = 4$ $$y(4) = \sum x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

For $n = 5$ $$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$$

$$\therefore \qquad y(n) = \left\{\begin{matrix} 4, 10, 13, 13, 10, 7, 3 \\ \uparrow \end{matrix}\right\}$$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 10$, $\sum_n h(n) = 6$ and $\sum_n y(n) = 60$

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved). Therefore, the result is correct.

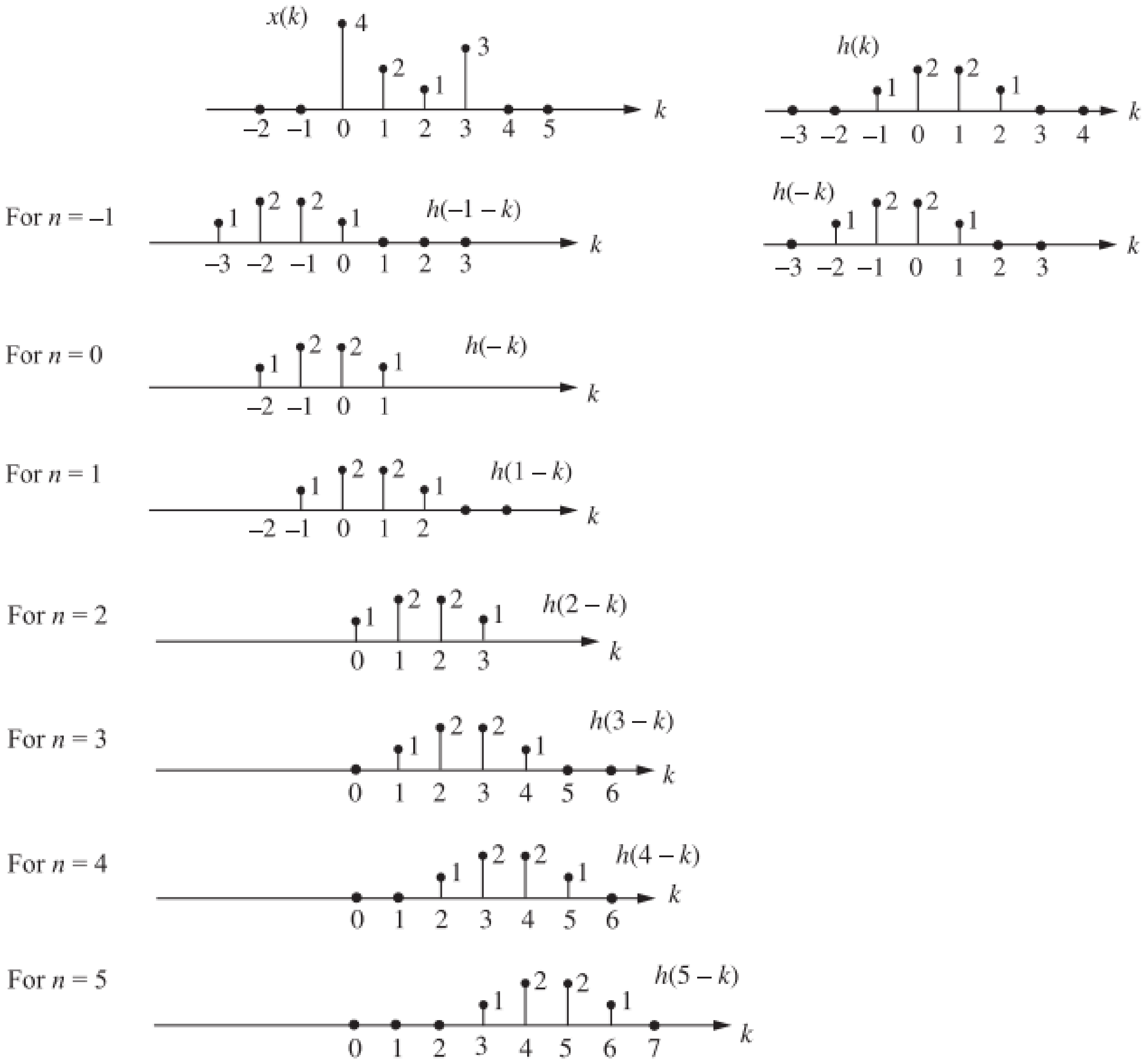


**Figure 2** Operation on signals $x(n)$ and $h(n)$ to compute convolution.

### *Method 2 Tabular array*

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 2

**TABLE 2** Table for computing $y(n)$.

| $k$ | | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x(k)$ | | – | – | – | – | 4 | 2 | 1 | 3 | – | – | – | – |
| $h(-k)$ | | – | – | 1 | 2 | 2 | 1 | – | – | – | – | – | – |
| $n = -1$ | $h(-1-k)$ | – | 1 | 2 | 2 | 1 | – | – | – | – | – | – | – |
| $n = 0$ | $h(-k)$ | – | – | 1 | 2 | 2 | 1 | – | – | – | – | – | – |
| $n = 1$ | $h(1-k)$ | – | – | – | 1 | 2 | 2 | 1 | – | – | – | – | – |
| $n = 2$ | $h(2-k)$ | – | – | – | – | 1 | 2 | 2 | 1 | – | – | – | – |
| $n = 3$ | $h(3-k)$ | – | – | – | – | – | 1 | 2 | 2 | 1 | – | – | – |
| $n = 4$ | $h(4-k)$ | – | – | – | – | – | – | 1 | 2 | 2 | 1 | – | – |
| $n = 5$ | $h(5-k)$ | – | – | – | – | – | – | – | 1 | 2 | 2 | 1 | – |

The starting value of $n = -1$. From the table, we can see that

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

For $n = 4$ $$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

For $n = 5$ $$y(5) = \sum x(k)h(5-k) = 3 \cdot 1 = 3$$

$$\therefore \quad y(n) = \left\{ \begin{matrix} 4, & 10, & 13, 13, 10, 7, 3 \\ & \uparrow & \end{matrix} \right\}$$

## *Method* 3 *Tabular method*

Given $x(n) = \{4, 2, 1, 3\}$, $h(n) = \{1, \underset{\uparrow}{2}, 2, 1\}$

The convolution of $x(n)$ and $h(n)$ can be computed as shown in Table 3.

**TABLE 3** Table for computing $y(n)$.

| $h(n)$ \ $x(n)$ | 4 | 2 | 1 | 3 |
|---|---|---|---|---|
| 1 | 4 | 2 | 1 | 3 |
| 2 | 8 | 4 | 2 | 6 |
| 2 | 8 | 4 | 2 | 6 |
| 1 | 4 | 2 | 1 | 3 |

$$y(n) = 4,\ 8+2,\ 8+4+1,\ 4+4+2+3,\ 2+2+6,\ 1+6,\ 3$$
$$= 4,\ 10,\ 13,\ 13,\ 10,\ 7,\ 3$$

The starting value of $n$ is equal to –1, mark the symbol ↑ at time origin ($n = 0$).

$$\therefore \qquad y(n) = \{4, \underset{\uparrow}{10}, 13, 13, 10, 7, 3\}$$

## *Method* 4 *Matrices method*

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{4, 2, 1, 3\}$

and $h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, \underset{\uparrow}{2}, 2, 1\}$

The sequence $x(n)$ is starting at $n = 0$ and the sequence $h(n)$ is starting at $n = -1$. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $h(n)$ will start at $n = 0 + (-1) = -1$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 - 1 = 7$. Substituting the sequence values in matrix form and multiplying as shown below, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 13 \\ 13 \\ 10 \\ 7 \\ 3 \end{bmatrix}$$

$$y(n) = x(n) * h(n) = [4\ \ \underset{\uparrow}{10}\ \ 13\ \ 13\ \ 10\ \ 7\ \ 3]$$